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STABILIZABILITY OF TRIVIAL STEADY MOTIONS OF GYROSCOPICALLY COUPLED SYSTEMS WITH PSEUDO-CYCLIC COORDINATES*

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A qualitative analysis is carried out of the problem of the stabilization of steady motions of systems with pseudo-cyclical coordinates /1—3/. Established trivial motions with a non-zero instability of gyroscopically uncoupled systems cannot be stabilized /3/. The property of gyroscopic connectedness of a system presents certain stabilization possibilities.

1. Suppose a mechanical system with r positional coordinates q_i and $n-r$ pseudo-cyclic coordinates is gyroscopically coupled, i.e. its kinetic energy has the form

$$T = \frac{1}{2} q^{\cdot T} A(q) q^{\cdot} + q^{\cdot T} C(q) \omega + \frac{1}{2} \omega^T B(q) \omega$$

where $q, q^{\cdot}, \omega$ are column-matrices of the positional coordinates of the positional and pseudo-cyclic velocities, A and B are positive definite symmetric matrices, and C is a rectangular matrix. Their coefficients depend on the positional coordinates.

Suppose that the generalized forces that correspond to positional coordinates are specified and represent the sum of potential and dissipative forces

$$Q_i = \partial U / \partial q_i - Q_{id}$$

The generalized forces F_j that correspond to pseudo-cyclic coordinates are assumed to be controlling and are subject to selection.

Let the system admit of the trivial stabilized motion

$$q_0 = \text{const} \tag{1.1}$$

which means that /3/

$$\frac{\partial U(q_0)}{\partial q_i} = \frac{\partial B_{kj}(q_0)}{\partial q_i} = 0 \quad (i, k, j = 1, \ldots, r) \tag{1.2}$$

The pseudo-cyclic velocities of stabilized motion (1.1) under condition (1.2) may be arbitrarily specified.

2. The question of the stabilizability of a trivial steady motion q_0, ω_0 to a first approximation reduces to the analysis of Lagrange's equations linearized in the neighbourhood of the point q_0, ω_0

$$A_i x^{\cdot\cdot} - C_i \eta^{\cdot} - G_i x^{\cdot} + D_i x^{\cdot} + W_i x = 0 \tag{2.1}$$

$$x^{\cdot\cdot T} C_j + B_j \eta^{\cdot} = K_j \eta + P_j x + N_j x^{\cdot} \tag{2.2}$$

$$x = q - q_0, \quad \eta = \omega - \omega_0, \quad G_i = \omega^T \left(\frac{\partial C_i}{\partial q} - \frac{\partial C}{\partial q_i} \right)$$

$$W_i = \frac{\partial^2 U}{\partial q_i \partial q} + \omega^T \frac{\partial^2 B}{\partial q_i \partial q} \omega, \quad i = 1, \ldots, r; \quad j = r+1, \ldots, n$$

Here D_i is the i-th row of matrix D (the linear part of the dissipative force Q_{id}; we assume that $\det D \neq 0$). A_i, C_i, B_j are the rows of the matrices A, C, B, and K_j, P_j, N_j are the rows of matrices K, P, N of linear controlling forces F_j. All the coefficients of the system are calculated for $q = q_0, \omega = \omega_0$.

Note that for gyroscopically uncoupled mechanical systems $C \not\equiv 0$, and the subsystem (2.1) "splits off" from subsystem (2.2) which contains the controlling forces. But in the case considered here of $C \equiv 0$ the subsystem (2.1) and (2.2) have a crossing connection via $\eta^{\cdot}$ and $x^{\cdot\cdot}$, that presents certain stabilization possibilities. Naturally, this problem is important first of all, when the matrix W is not positive definite. In particular, if W has l negative eigenvalues (the degree of instability equals l), the zero solution of subsystem (2.1) with condition $\eta^{\cdot} \equiv 0$ is unstable (by virtue of the Kelvin-Chetayev theorem).

To construct the stabilizing actions we will use the method described in /3/. We shall

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stipulate that system (2.1) shall possess an asymptotically invariant manifold

$$\eta + Lx + Mx^{\cdot} = 0. \tag{2.3}$$

For this it is sufficient that the system

$$y^{\cdot} = -\gamma y, \quad y = \eta + Lx + Mx^{\cdot} \tag{2.4}$$

be identically satisfied by virtue of (2.1) and (2.2). In (2.4) γ is a symmetric positive definite matrix. The selection of the matrices L, M, γ uniquely defines the coefficients of the matrices K, P, N of the controlling forces.

The system (2.1), (2.2) has in the invariant manifold (2.3) the form

$$\begin{aligned} &A_* x^{\cdot\cdot} + [Q_*(D+G) + CB^{-1}S_*^{-1}L]\, x^{\cdot} + Q_* W x = 0 \\ &A_* = A - CB^{-1}C^T, \quad B_* = B - C^T A^{-1} C \\ &S_* = -B_*^{-1} + MA_*^{-1}CB^{-1}, \quad T_* = -B_*^{-1}C^T A^{-1} + MA_*^{-1}, \\ &Q_* = E - CB^{-1}S_*^{-1}T_* \end{aligned} \tag{2.5}$$

where the matrices G, W are formed by rows G_i, W_i).

The following statement holds: if the selection of the matrices L, M is such that the zero solution in system (2.5) becomes asymptotically stable, the steady motion $q = q_0$, $\omega = \omega_0$ is stabilizable. The coefficients of the matrices of the controlling forces that stabilize the established motion are given by the formulas

$$\begin{aligned} &K = S_*^{-1}\gamma, \quad P = S_*^{-1}(\gamma L - T_* W) \\ &N = S_*^{-1}[\gamma M + L - T_*(D+G)] \end{aligned} \tag{2.6}$$

The system (2.5) is of a general form. Its dimensions are determined only by the number of positional coordinates.

Let us assume that the degree of instability l does not exceed the number of pseudo-cyclic coordinates $(n-r)$. It then becomes possible to select the matrix M, so that the matrix $Q_* W$ is symmetric and positive definite. Then by a suitable selection of the matrix L we can make the symmetric part of the matrix composed of the coefficients of $x^{\cdot}$ in (2.5) positive definite. The matrices M and L selected in this way solve the problem of constructing the stabilized actions /4/.

3. Let us illustrate the above on the problem of the stabilization of the steady trivial motions of a heavy gyroscope in gimbals with directed violation of symmetry and the vertical axis of rotation of the outer ring.

Investigations of the steady motions of a perfect gyroscope are described in a number of papers (/3, 5/ and others). The term perfect gyroscope means a gyroscope whose inner ring axis is orthogonal to both the outer ring and the rotor axes, with all these axes intersecting at one point, and each of these is the principal axis of inertia of the respective body.

Consider a system consisting of three rigid bodies: S^1 the outer ring and S^2 the inner ring, respectively, and S^3 the rotor located in the gravity field (Fig.1). The body S^1 is connected to the stationary base by means of a cylindrical hinge with axis l^1 in the direction of the force of gravity. The bodies S^1 and S^2 are connected to each other by a cylindrical link with the l^2 axis intersecting the l^1 axis at the point O. We assume the rotor axis l^3 is fixed in the body and the rotor masses are symmetrically distributed about the axis. The centre of mass of the rotor is denoted by O_0.

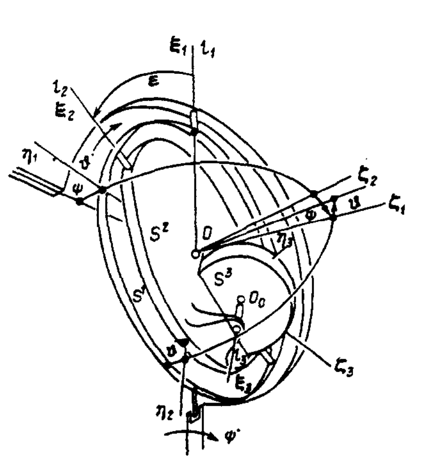

Fig.1

We select $O\xi_1\eta_1\zeta_1$, $O\xi_2\eta_2\zeta_2$ and $O_0\xi_3\eta_3\zeta_3$ as the moving systems of coordinates. The axes $O\xi_1$, $O\xi_2$, $O_0\xi_3$ coincide with l^1, l^2, l^3, respectively. The plane $O\xi_1\eta_1$ contains the axis $O\xi_2$. We denote the angle between $O\xi_1$ and $O\xi_2$ by ε $(0 < \varepsilon < \pi)$; λ, μ, ν are the direction cosines of the rotor axis l^3 in the system $\xi_2\eta_2\zeta_2$.

The position of the system of bodies in space is defined by the Euler angles: ψ is the angle of rotation of the outer ring, the angle of precession, ϑ is the angle of rotation of the inner ring relative to the outer, the angle of nutation, and φ is the angle of rotation of the rotor about the l^3 axis relative to the inner gimbal ring. We assume that $\vartheta = 0$, when the l^3 axis is in a plane parallel to l^1 and l^2 $(0 \leqslant \vartheta \leqslant \pi)$.

We introduce the following notation: J is the moment of inertia of the outer gimbal ring about the l^1 axis,

$$I = \begin{Vmatrix} A_2 & -G_2 & -R_2 \\ -G_2 & B_2 & -E_2 \\ -R_2 & -E_2 & C_2 \end{Vmatrix}$$

is the tensor of inertia of the inner gimbal ring in axes $O\xi_2\eta_2\zeta_2$. A_0 and B_0 are respectively the axial and equatorial moments of inertia of the rotor about the point O_0, m_2, m_3 are the masses of the inner gimbal ring, x_1, y_1, z_1 are the coordinates of the rotor centre of mass O_0 and x_2, y_2, z_2 are the coordinates of the common mass of the inner gimbal ring and rotor in the system $O\xi_2\eta_2\zeta_2$.

The parameters of the gyroscope are assumed to satisfy the following conditions:

$\nu = 0$ (the rotor axis l^3 is parallel to plane $O\xi_2\eta_2$ and coupled to the inner gimbal ring;

$z_2 = 0$ (the common centre of mass of the inner gimbal ring and rotor is in the plane $O\xi_2\eta_2$);

The distribution of masses in the bodies is such that the equations

$$E_2 + m_3 y_1 z_1 = 0, \quad R_2 + m_3 z_1 x_1 = 0$$

are satisfied.

The generalized force that corresponds to the positional coordinate ϑ, represent the sum of the moment of the force of gravity with force function U and the dissipative force $d\vartheta^{\cdot}$.

The control forces are the actions of the moment F_1 of the motor rotating the outer gimbal ring and of the moment F_2 of the motor on the inner gimbal ring and rotating the gyroscope rotor.

The kinetic energy and force function of the mechanical system considered here have the form /6, 7/

$$T = {}^1/_2\, [a\vartheta^{\cdot 2} - 2(c_1\omega_1 + c_2\omega_2)\,\vartheta^{\cdot} + b_{11}\omega_1^2 + 2b_{12}\omega_1\omega_2 + b_{22}\omega_2^2]$$
$$\psi^{\cdot} = \omega_1, \quad \varphi^{\cdot} = \omega_2; \quad U = (m_2 + m_3)\, g y_2 \sin\varepsilon\cos\vartheta$$
$$a = A_2 + B_0 + (A_0 - B_0)\,\lambda^2 - (y_1^2 + z_1^2)\, m_3$$
$$c_1 = [A_2 - B_0 + (A_0 - B_0)\,\lambda^2 + m_3\,(y_1^2 - z_1^2)]\cos\varepsilon + [G_2 + (B_0 - A_0)\,\lambda\mu - m_3 x_1 y_1]\sin\varepsilon\cos\vartheta, \quad c_2 = A_0\lambda$$
$$b_{11} = J + [A_2 - B_0 + (A_0 - B_0)\,\lambda^2 + m_3\,(y_1^2 - z_1^2)]\cos^2\varepsilon + {}^1/_2\,[B_2 - C_2 + 2B_0 - (A_0 - B_0)\,\mu^2 + m_3\,(z_1^2 - 2x_1^2 + y_1^2)]\sin^2\varepsilon + [G_2 + (B_0 - A_0)\,\lambda\mu + m_3 x_1 y_1]\sin 2\varepsilon\cos\vartheta + {}^1/_2\,[B_2 - C_2 + (A_0 - B_0)\,\mu^2 + m_3\,(z_1^2 - y_1^2)]\sin^2\varepsilon\cos 2\vartheta$$
$$b_{12} = A_0\lambda\cos\varepsilon - A_0\mu\sin\varepsilon\cos\vartheta, \quad b_{22} = A_0$$

It can be shown that condition (1.2) is satisfied when $\sin\vartheta = 0$.

Thus among the steady motions of a gyroscope there are trivial ones $\vartheta = 0$ and $\vartheta = \pi$. Note that, unlike the traditional case /3/, the gyroscope rotor in the trivial motion performs a regular precession (the angle between the stationary axis l^1 and the rotor axis is non-zero), and the angular velocities of prcession ω_1 and of the proper rotation ω_2 may be arbitrary.

We restrict the investigation to the motion $\vartheta = 0$, $\omega_1 = \text{const}$, $\omega_2 = \text{const}$. The equations in deviations (2.1), (2.2) have the form

$$ax^{\cdot\cdot} + c_1\eta_1^{\cdot} + c_2\eta_2^{\cdot} - dx^{\cdot} + Wx = 0$$
$$c_1 x^{\cdot\cdot} + b_{11}\eta_1^{\cdot} + b_{12}\eta_2^{\cdot} = K_1\eta_1 + K_2\eta_2 + N_1 x^{\cdot} + P_1 x$$
$$c_2 x^{\cdot\cdot} + b_{12}\eta_1^{\cdot} + b_{22}\eta_2^{\cdot} = K_3\eta_1 + K_4\eta_2 + N_2 x^{\cdot} - P_2 x$$
$$W = {}^1/_2\,\{[G_2 + (B_0 - A_0)\,\lambda\mu - m_3 x_1 y_1]\sin 2\varepsilon + 2\,[B_2 - C_2 - (A_0 - B_0)\,\mu^2 - m_3\,(z_1^2 - y_1^2)]\sin^2\varepsilon\}\,\omega_1^2 - A_0\mu\sin\varepsilon\,\omega_1\omega_2 - (m_2 + m_3)\, g y_2 \sin\varepsilon$$

where $K_1, K_2, K_3, K_4, N_1, N_2, P_1, P_2$ are the coefficients of the controlling forces F_1, F_2.

Consider the case $W < 0$ which occurs when $\omega_1 = \omega_2 = 0$ and $y_2 < 0$, when the gyroscope is in such equilibrium that its centre of mass is above the point of suspension. Without control actions that equilibrium is unstable.

Equation (2.5) in that case has the form

$$x^{\cdot\cdot} + (\Delta_1/\Delta)\, x^{\cdot} + (W/\Delta)\, x = 0$$
$$\Delta_1 = d - c_1 L_1 - c_2 L_2, \quad \Delta = a - c_1 M_1 - c_2 M_2$$

It is obvious that for stabilization it is sufficient to select the coefficients L_1, L_2, M_1, M_2 of the matrices L, M so that the inequality $\Delta_1 < 0$, $\Delta < 0$ is satisfied.

The coefficients of the matrices of the control forces are calculated from (2.6).

Since the mechanical system considered here has only one positional coordinate and two pseudo-cyclical coordinates, there is some arbitrariness in the selection of L_1, L_2, M_1, M_2. These coefficients may, in particular, be selected so that the control actions will depend either on the positional coordinate or on the positional velocity.

The property of gyroscopic connectedness thus provides new qualitative possibilities in the problem of stabilizing steady motions.

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NORMAL OSCILLATIONS OF A STRING WITH CONCENTRATED MASSES ON NON-LINEARLY ELASTIC SUPPORTS*

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The problem of constructing and classifying normal oscillations of a string with concentrated masses on non-linearly elastic supports is considered (special limit cases are linear and vibro-impact systems). It is shown that in the limit of intensive impact action, the non-linear system has supplementary properties of symmetry which enables this problem to be solved effectively. On the other hand, the normal oscillations of a vibro-impact system can be used as the generating solutions for dynamic calculations of essentially non-linear systems that are close to them. The connection between localized normal oscillations and solutions of the soliton type are discussed.

The existence of normal oscillations as special particular solutions of linear conservative system is due to the properties of symmetry inherent in it that can be partly retained in the non-linear case /1, 2/. The possibility of constructing in some strongly non-linear system synchronous motions that have a number of properties of normal linear oscillations /3, 4/ is related to this property. The normal "principal" oscillations have already been considered by Lyapunov /5/. Due to these properties a multidimensional non-linear system in the normal oscillation mode behaves as a non-linear oscillator.

1. Consider the transverse oscillations of a regular chain of masses connected to each other by a weightless string, and interacting with non-linearly elastic supports (Fig.1) which in the limit become rigid limiters with some gap $2e$. The reaction of the j-th support is

$$q_j = a\,(u_j/e)^{2n-1}$$

where u_j is the deflection of the string in the respective cross section, n is an integer, and a is the stiffness parameter.

Among the various type of motions of non-linear systems are those of the simplest mode, such as normal oscillations. When $n = 1$ (a string with concentrated masses on linearly elastic supports) we have N normal oscillations; the spectrum of respective natural frequencies is discrete and limited.

When $n > 1$ we distinguish the cases of strong and weak connection along the string. In the first case the non-linearity is small and the system belongs to the Lyapunov class of systems /5/. We thus have the problem of constructing non-linear normal oscillations which, as the amplitude decreases, become normal oscillations of a normalized system. This simple case is not considered further.

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